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Space/time non-commutativity and causality

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Abstract: Field theories based on non-commutative spacetimes exhibit very distinctive non-local effects which mix the ultraviolet with the infrared in bizarre ways. In particular if the time coordinate is involved in the non-commutativity the theory seems to be seriously acausal and inconsistent with conventional hamiltonian evolution. To illustrate these effects we study the scattering of wave packets in a field theory with space/time non-commutativity. In this theory we find effects which seem to precede their causes and rigid rods which grow instead of Lorentz contract as they are boosted. These field theories are evidently inconsistent and violate causality and unitarity. On the other hand open string theory in a background electric field is expected to exhibit space/time non-commutativity. This raises the question of whether they also lead to acausal behavior. We show that this is not the case. Stringy effects conspire to cancel the acausal effects that are present for the non-commutative field theory.

Keywords: Superstrings and Heterotic Strings, Non-Commutative Geometry.

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Contents

1. Introduction

Non-commutative field theory is a model of a world with non-commuting spatial coordinates (space/space non-commutativity) and in fact such field theories do arise as the description of string theory in certain backgrounds [1, 2, 3]. Non-commutative theories are very non local in the non-commuting spatial directions but are quadratic in time derivatives. Nonlocality in spatial directions ruins Lorentz invariance but it is consistent with the basic rules of hamiltonian quantum mechanics. Action at a distance may occur but events never precede their causes.

The situation is much less clear for field theories with non-commutativity between time and a space direction (space/time non-commutativity). The action is arbitrarily non local in time with the evolution of fields at one time depending on the value of fields at both past and future times. The question then is whether the kind of unusual behavior found in space/time non-commutative field theories can ever occur in any consistent theory with a hamiltonian and a unitary S-Matrix. We don't know the answer to this question but we examine an obvious candidate, string theory in a background electric $B_{\mu\nu}$ field. This theory is manifestly unitary and may be expected to exhibit effects similar to those seen in the field theory example. However as we have seen in [4, 5] the theory in an electric field never becomes a field theory and retains its stringy excitations. We will see that the stringy effects cancel the acausal effects of space/time non-commutativity.

The plan of the paper is as follows. In section 2 we describe the scattering of wave packets in non-commutative field theory. In the case of space/space non-commutativity the scattering induces a sudden spatial displacement of the wave packet in the direction orthogonal to the momentum. The magnitude of displacement is proportional to the momentum. As expected the degree of nonlocality increases with momentum. The scattered particles behave like rigid rods oriented perpendicular to their momentum with a size proportional to their momentum [3, 6, 7, 8].

In the space/time case the wave packets scatter in a manner that appears to violate causality. Two scattered packets appear. One of them is physically sensible and corresponds to a time delay proportional to the incoming momentum. The other wave packet has a negative time delay! As in the space/space case the effect increases with the momentum. Thus at very high energy one of the outgoing waves appears to originate long before the particles could have collided. We refer to this behavior as *advanced*. Alternatively the particles behave like rods oriented along the direction of motion. Again the length of the rod is proportional to its momentum. This increase of length with momentum is very counterintuitive and is quite opposite to the expected Lorentz contraction.

Having defined the distinctive signatures of space/time non-commutativity, in section 3 we proceed to look for string theory realizations of these signatures. We investigate open string scattering with and without background electric fields. We find delayed wave packets with time delay proportional to momentum as expected in non-commutative theories. We emphasize that the delayed effect occurs *with and without* a background electric field. However in no case do we find the acausal signatures of space-time non-commutativity. The case of open strings in electric fields is particularly interesting. Although the amplitudes acquire Moyal phases the stringy effects mask the phases that would otherwise give rise to advanced effects. The scattering with the electric field does not seem appreciably different than without it even in the critical limit.

2. Scattering in non-commutative field theory

In this section we study the effect of space/space and space/time non-commutativity on the scattering of massless scalar particles. We begin with the space/space case. To illustrate the main points it is sufficient to consider $(2+1)$-dimensional non-commutative scalar ϕ^4 theory in lowest-order perturbation theory. The coordinates are labeled (x, y, t). Since this case is familiar we will just describe the scattering schematically. Let us consider two high energy particles moving along the x axis with spatial momentum P_x. We will take the initial wave function to be

$$\Psi(x, y) = \exp(iP_x x)\psi_{\rm in}(y)\,, \tag{2.1}$$

where

$$\psi_{\rm in}(y) = \int dP_y \hat{\psi}_{\rm in}(P_y) e^{iP_y y}\,. \tag{2.2}$$

The important feature of the scattering amplitude for our purposes is the Moyal phase factors which take the form

$$M = \exp \frac{i}{2}\theta(P_x Q_y - P_y Q_x)\,, \tag{2.3}$$

where Q is the momentum transfer. We assume $P_x \gg P_y, Q_x, Q_y$.

After the scattering, the scattered momentum space wave function is given by an expression of the form

$$\hat{\psi}_{\rm out}(P_y) = \int dQ_y \hat{\psi}_{\rm in}(P_y + Q_y) \exp \frac{i}{2}\theta(P_x Q_y)\,. \tag{2.4}$$

In coordinate space

$$\psi_{\rm out}(y) \approx \psi_{\rm in}(y)\delta\left(y - \frac{1}{2}\theta P_x\right)\,. \tag{2.5}$$

In other words the outgoing scattered wave appears to originate from the displaced position $y = \theta P_x/2$.

An intuitive way to understand this effect is to think of the incident particles as extended rods oriented perpendicular to their momentum [3, 6, 7, 8]. The size of the rods is θP and the rule is that they only interact if their ends touch.

Now we turn to the more interesting case of space/time non-commutativity which we will study in much more detail. For simplicity we will work in $1+1$ dimensions. We denote time by t and the spatial variable by x.

Let us begin by reviewing the scattering of wave packets in 1+1 dimensions. A free scalar field in $1+1$ dimensions has the following Fourier decomposition

$$\phi(x,0) = \int \frac{dp}{(2\pi)\sqrt{2E_p}} \left(a_p e^{ipx} + a_p^\dagger e^{-ipx}\right) \tag{2.6}$$

with

$$\left[a_p, a_k^\dagger\right] = (2\pi)\delta(p-k)\,. \tag{2.7}$$

Because of the special infrared divergences of massless $(1+1)$-dimensional scalar fields we will work with the derivative of ϕ rather than ϕ itself:

$$\phi'(x,0) = i\int \frac{dp}{(2\pi)\sqrt{2E_p}} \left(p a_p e^{ipx} - p a_p^\dagger e^{-ipx}\right)\,. \tag{2.8}$$

Single particle states with momentum p are normalized as follows:

$$|p\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle\,. \tag{2.9}$$

Then the norm

$$\langle p|k\rangle = 2E_p(2\pi)\delta(p-k) \tag{2.10}$$

is Lorentz invariant. The wavefunction of such a state will be defined by

$$\langle 0|\phi'(x)|p\rangle = ipe^{ipx} . \tag{2.11}$$

Using the equation of motion for the free scalar field, we can find the wavefunction at all times.

Next, we turn on some interactions. For example, consider a commutative ϕ^4 interaction. We are interested in the scattering of massless scalars, in particular 2-body to 2-body scattering. For sufficiently high energies, we can use perturbation theory to calculate an S-matrix. The S-matrix takes the following form

$$S = 1 + iT , \tag{2.12}$$

where

$$\langle p_1, p_2|iT|k_1, k_2\rangle = (2\pi)^2\delta^2(k_1 + k_2 - p_1 - p_2)i\mathcal{M}(k_1, k_2 \to p_1, p_2) . \tag{2.13}$$

Here, k_1, k_2 denote the 2-momenta of the incoming particles and p_1, p_2 the 2-momenta of the outgoing particles. The invariant amplitude $i\mathcal{M}$ is computed in the usual way using Feynman diagrams. For the simple case of a ϕ^4 interaction,

$$i\mathcal{M} = -ig \tag{2.14}$$

to leading order in perturbation theory. In $1 + 1$ dimensions the only effect one expects to see in 2-body to 2-body scattering is time delays.

Now, consider an incoming state consisting of correlated pairs of particles with opposite momenta:

$$|\phi\rangle_{\rm in} = \int \frac{dk}{(2\pi)2E_k}\phi_{\rm in}(k)|k, -k\rangle , \tag{2.15}$$

with

$$\phi_{\rm in}(k) = \phi_{\rm in}(-k) . \tag{2.16}$$

The wavefunction of such a state is given by

$$\Phi_{\rm in}(x) \equiv \langle 0|\phi'(x_1)\phi'(x_2)|\phi\rangle_{\rm in} = 2\int \frac{dk k^2}{(2\pi)2E_k}\phi_{\rm in}(k)e^{ikx} , \tag{2.17}$$

where $x = x_1 - x_2$ is the relative separation of the two particles. There is no dependence on the center of mass position, since the overall center of mass momentum is zero. Let us also choose $\phi_{\rm in}(k)$ so that at the time of the collision $t = 0$, the wave-packet is well concentrated at $x = 0$. Then the incoming particles are close together at $t = 0$. For example, we may choose

$$\phi_{\rm in}(k) = e^{-(k-k_0)^2/\lambda} + e^{-(k+k_0)^2/\lambda} . \tag{2.18}$$

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The wavepacket is concentrated at energies closed to k_0. The width of the packet in space is given by $1/\lambda^{1/2}$. We let $\lambda \ll k_0^2$ and take k_0 large. At earlier times, $t<0$, we can use the free equations of motion to find that the packet is concentrated at $x=2t$. This means that the incoming particles are far apart in the past and they collide at $t=0$.

Similarly, the outgoing state is taken to be

$$|\phi\rangle_{\text{out}} = \int \frac{dp}{(2\pi)2E_p}\phi_{\text{out}}(p)|p,-p\rangle\,. \tag{2.19}$$

Then,

$$|\phi\rangle_{\text{out}} = S|\phi\rangle_{\text{in}} = |\phi\rangle_{\text{in}} + iT|\phi\rangle_{\text{in}}\,. \tag{2.20}$$

Therefore, we have that

$$\frac{\phi_{\text{out}}(p)}{(2\pi)2E_p} = \frac{\phi_{\text{in}}(p)}{(2\pi)2E_p} + \frac{\langle p,-p|iT|\phi\rangle_{\text{in}}}{8(2\pi)^2E_p^2\delta(0)}\,. \tag{2.21}$$

Now, using the form of the matrix element $\langle p,-p|iT|k,-k\rangle$, eq. (2.13), we find that the non-trivial part of $\phi_{\text{out}}(p)$ is given by

$$\int \frac{dk}{(2\pi)2E_k}\phi_{\text{in}}(k)\left(\frac{i\mathcal{M}}{8E_p^2}\right)\delta(2E_k-2E_p) = \frac{\phi_{\text{in}}(p)}{(2\pi)2E_p}\frac{i\mathcal{M}}{8E_p^2}\,. \tag{2.22}$$

Therefore, using eq. (2.17), the non-trivial part of the outgoing wavefunction can be obtained by

$$\Phi_{\text{out}}(x) \equiv \langle 0|\phi'(x_1)\phi'(x_2)|\phi\rangle_{\text{out}} = 2\int \frac{dpp^2}{(2\pi)2E_p}\phi_{\text{in}}(p)\frac{i\mathcal{M}}{8E_p^2}e^{ipx}\,. \tag{2.23}$$

In the case of the ϕ^4 theory, we see that nothing much happens. Choose $\phi_{\text{in}}(p)$ to be a polynomial in p times a gaussian so that the integral converges. Then $\Phi_{\text{in}}(x)$ is concentrated at $x=0$. Since $i\mathcal{M}\sim g$, at time $t=0$, the outgoing wavefunction will also be concentrated at $x=0$. Therefore, there are no large time delays. Using the free equations of motion, we find that at later times the wave-packet is concentrated at $x=2t$ and so the outgoing particles separate in the far future.

Consider now the effect of space/time non-commutativity.

$$[t,x] = i\theta\,. \tag{2.24}$$

The theory is defined by replacing the ordinary product by a $*$-product given by

$$\phi_1 * \phi_2(x,t) = e^{i\frac{\theta}{2}[\partial_0^y\partial_1^z - \partial_1^y\partial_0^z]}\phi(y)\phi(z)|_{y=z=(x,t)}\,. \tag{2.25}$$

The ϕ^4 lagrangian contains now an infinite number of time derivatives from the interaction term

$$g\phi*\phi*\phi*\phi\,. \tag{2.26}$$

Therefore the theory is not local in time. It is not clear that such a theory has a well defined hamiltonian. One plus one-dimensional Lorentz invariance however, is undisturbed by the non-commutativity. This is easily seen from the fact that the defining commutation relation has the form

$$[x^\mu, x^\nu] = i\theta\epsilon^{\mu\nu}\,. \tag{2.27}$$

The effect of the $*$-product is to produce phases in the interaction vertex that depend on the energies of the particles. The tree-level scattering amplitude is now given by

$$i\mathcal{M} \sim g\Big[\cos(p_1\wedge p_2)\cos(p_3\wedge p_4) + 2\leftrightarrow 3 + 2\leftrightarrow 4\Big]\,, \tag{2.28}$$

where p_1, p_2, p_3, p_4 are the 2-momenta of the particles satisfying

$$p_1 + p_2 + p_3 + p_4 = 0\,. \tag{2.29}$$

Here $p\wedge k = \theta(p^0 k^1 - k^0 p^1)$. Note that we have used conventions with all particles taken to be incoming in the vertices; i.e. energies of outgoing particles are negative. In the center of mass frame with the incoming particles (and outgoing) having equal and opposite spatial momenta, the amplitude becomes

$$i\mathcal{M} \sim g\left[\cos(4p^2\theta) + 2\right]\,. \tag{2.30}$$

The pattern is similar in more general non-commutative theories but depending on the spins and polarizations of the particles the periodic functions may be sines in place of cosines.

We remark that such a theory fails to be unitary at the 1-loop level [9]. However, let us just consider tree-level scattering amplitudes and in particular the effect of non-commutativity on the outgoing wave-packets. We choose for the incoming wave-packet $\phi_{\rm in}(p)$ a gaussian function:

$$\phi_{\rm in}(p) \sim E_p\left(e^{-(p-p_0)^2/\lambda} + e^{-(p+p_0)^2/\lambda}\right)\,. \tag{2.31}$$

(The extra factor of E_p is added to simplify the integrals but does not change the qualitative behavior of our results.) Using eq. (2.19), we can find the outgoing wavefunction

$$\Phi_{\rm out}(x) \sim g\int dp\left[\cos(4p^2\theta) + 2\right]\left(e^{-(p-p_0)^2/\lambda} + e^{-(p+p_0)^2/\lambda}\right)e^{ipx}. \tag{2.32}$$

To compute the integral, we need to calculate the following Fourier transform:

$$\int dp e^{4ip^2\theta} e^{-(p-p_0)^2/\lambda} e^{ipx} \sim e^{-p_0^2/\lambda}\frac{1}{\sqrt{1/\lambda - 4i\theta}} e^{(2p_0/\lambda + ix)^2\frac{1}{4(1/\lambda-4i\theta)}} = \tag{2.33}$$

$$= \frac{1}{\sqrt{1/\lambda - 4i\theta}}\exp\left[\frac{-\lambda\left(x + 8p_0\theta\right)^2}{4(1+16\theta^2\lambda^2)}\right]\exp\left[\frac{-i\theta\lambda^2\left(x - \frac{p_0}{2\lambda^2\theta}\right)^2}{1+16\theta^2\lambda^2}\right]\exp i\frac{p_0^2}{4\lambda^2\theta}\,.$$

We take $p_0 \gg \lambda^{1/2} \gg 1/p_0\theta$, and also assume that $\lambda\theta \gg 1$. Then, eq. (2.33) simplifies as follows:

$$\frac{1}{\sqrt{-4i\theta}} e^{-\frac{(x+8p_0\theta)^2}{64\theta^2\lambda}} e^{-i\frac{(x-\frac{p_0}{2\lambda^2\theta})^2}{16\theta}} e^{i\frac{p_0^2}{4\lambda^2\theta}} \equiv F(x;\theta,\lambda,p_0)\,. \tag{2.34}$$

Then the outgoing wavefunction is given by

$$\Phi_{\rm out}(x) \sim g\Big[F(x;-\theta,\lambda,p_0) + 4\sqrt{\lambda}e^{-\lambda x^2/4}e^{ip_0x} + \\ + F(x;\theta,\lambda,p_0)\Big] + (p_0 \to -p_0)\,. \tag{2.35}$$

We see that the wave-packet splits into three parts, one concentrated at $x = 8p_0\theta$, one at $x = 0$ and the other at $x = -8p_0\theta$. The width of the first and third packet is given by $8\lambda^{1/2}\theta$ while the one concentrated at $x = 0$ has width $2/\lambda^{1/2}$. Therefore, the packets are well separated for $p_0 \gg \lambda^{1/2} \gg 1/p_0\theta$. The separation of the two displaced packets is proportional to p_0 which is the energy of the particles. The bigger the energy is the bigger the separation.

The packet at $x = 0$ oscillates with frequency p_0. The other two packets oscillate with phases $\exp[i(x + p_0/2\lambda^2\theta)^2/16\theta]$ and $\exp[-i(x - p_0/2\lambda^2\theta)^2/16\theta]$. Locally, near the maxima at $x = \pm 8p_0\theta$ the phases in the other two packets become $\exp[ip_0(1 + 1/16\lambda^2\theta^2)\Delta x]$ and so they oscillate with frequency p_0 since $\lambda\theta \gg 1$. This was expected from energy conservation.

All three wavepackets propagate towards $x \to \infty$. They correspond to particles 3 and 4 moving apart. In our conventions, particle 4 has momentum opposite to that of particle 1. We can think of it as particle 1 back-scattered. The first packet is an advanced wave. It appears at $x = 0$ at some time before the incoming wave arrives at the origin. The phase responsible for the acausal behavior is $e^{-4i\theta p^2}$. The third packet is delayed. The opposite phase causes the delay. Similarly the terms we get from $p_0 \to -p_0$ are waves moving towards $x \to -\infty$. It is easy to check that the advanced wave is again produced by the phase $e^{-4i\theta p^2}$.

Thus the collision is described as follows: the center of mass back scattering is isomorphic to bouncing off a wall. An incoming wave packet of spatial width $\lambda^{-1/2}$ is arranged to arrive at the wall at time $t = 0$. The outgoing wave consists of three terms. One term appears to originate from the wall at time $t = 8p_0\theta$, well after the incoming packet reached the wall. It is odd that the wave is delayed for so long a time as the energy increases but it is not acausal. A second term is neither significantly delayed or advanced. We will ignore it.

The other term is an "advanced" wave which appears to leave the wall *before* the incoming packet arrived. What is worse, the effect increases with energy so that the advance is proportional to the energy. This certainly seems acausal.

In itself, an advance does not violate causality. A simple non-relativistic model illustrates the point. Picture the incoming particles as rigid rods of length L. Assume the rod reflects when its leading end strikes the wall. In this case the center of mass of the rod will appear to reflect before it reaches the wall. In a newtonian world a physicist measuring an advance would conclude that the scattering objects resembled rigid rods.

The problem with such rigid rods is that they conflict with the combined constraints of causality and Lorentz invariance. In fact the required properties of the rod are completely at variance with the usual expectations of special relativity. For example one usually assumes that perfectly rigid bodies can not exit. The reason is that by suddenly displacing one end of a rod, the signal would instantly appear at the other end. Since such a rod is spacelike, this is usually thought to lead to action at a distance, nonlocality and violation of causality.

Equally peculiar is the behavior of the rods under boost. Suppose the momentum is increased. The conventional expectation is that the rod will Lorentz contract thus decreasing the advance. This is precisely the opposite of what space/time non-commutativity implies. The rod seems to expand as its momentum increases.

Another phenomenon predicted by eq. (2.35) is that the outgoing packet is much broader than the incoming. Let the incoming packet be of spatial width $\lambda^{-1/2}$. By contrast, the outgoing packet has spatial width $\lambda^{1/2}\theta$. In the limit we study of large $\lambda\theta$ this is broader than the incoming packet. How is this explained?

To understand this effect we return to the rod model. The advance is of order the rod size L. If we take $L = p\theta$ then the uncertainty in the rod size is

$$\Delta L = \theta \Delta p = \theta \lambda^{1/2} . \tag{2.36}$$

This means that the advance is also uncertain by the same amount. This obviously broadens the outgoing packet by the required amount.

All three terms in eq. (2.35) can be interpreted in terms of the rod model. Each of the incoming rods has two ends, a leading and a trailing end. The advanced term is due to the scattering of the two leading ends while the retarded contribution originates from the interaction of the trailing ends. The interaction of a leading and a trailing end contributes the second term in eq. (2.35).

What are we to make out of this behavior? The most obvious response is to dismiss it as pathological and declare space/time non-commutativity to be unphysical. Our opinion is that this is prematurely pessimistic. The main reason is that some of the properties of the amplitude largely follow from the uncertainty principle implied by eq. (2.24)

$$\Delta t \Delta x \geq \theta . \tag{2.37}$$

This uncertainty principle has the same form as the stringy uncertainty principle [10, 11, 12]

$$\Delta t \Delta x \geq \alpha' . \tag{2.38}$$

It therefore behooves us to inquire into the structure of string theory amplitudes to see if they produce any behavior similar to what we find in theories with space/time non-commutativity.

3. Scattering in open string theory

In this section, we analyze tree level scattering amplitudes of open strings on branes in the presence of a background electric $B_{\mu\nu}$ field. In the presence of a background electric field the underlying spacetime is non commutative. It is interesting to ask how scattering experiments similar to those studied in the previous example can probe the space/time non-commutativity. In particular we would like to investigate whether the acausal behavior we found in the simple field theory model is present. As we shall see, the amplitudes produce causal behavior and exhibit large time delays proportional to the momentum. The later phenomenon persists even without the electric field. The reader may think of the problem in a $(1+1)$-dimensional context by considering open string scattering on a stack of D1-branes [13, 14]. Throughout this section, we denote by g_s, l_s the string coupling constant and length scale.

At the level of disc amplitudes the inclusion of the electric field is simple. All we need is to start with the amplitudes at $E=0$, replace the metric $\eta_{\mu\nu}$ by the effective open string metric $G_{\mu\nu}$, g_s by G_s and multiply the answer by the phase factors with non-commutativity parameter θ. In terms of the electric field these parameters are given by [4]

$$G_{\mu\nu} = (1-\tilde{E}^2)\eta_{\mu\nu}\,, \quad \mu\nu = 0,1\,, \qquad G_{\mu\nu} = \delta_{\mu\nu}\,, \quad \mu\nu \neq 0,1\,,$$
$$\theta^{01} = 2\pi l_s^2 \frac{\tilde{E}}{1-\tilde{E}^2}\,, \qquad G_s = g_s(1-\tilde{E}^2)^{1/2}\,. \tag{3.1}$$

Here, $\tilde{E} = E/E_{\rm cr} \leq 1$. The critical electric field is given by $E_{\rm cr} = 1/2\pi l_s^2$.

Let us consider the Veneziano amplitude describing massless open string scattering. In terms of open string parameters the amplitude has the following form:

$$\begin{aligned} A_4 \sim\; & G_s \left(K_{st} e^{i(p_1\wedge p_2 + p_3\wedge p_4)} + K'_{st} e^{i(p_1\wedge p_4 + p_3\wedge p_2)}\right) \frac{\Gamma(-2sl_s^2)\Gamma(-2tl_s^2)}{\Gamma(1+2ul_s^2)} + \\ & + G_s \left(K_{su} e^{i(p_1\wedge p_2 + p_4\wedge p_3)} + K'_{su} e^{i(p_1\wedge p_4 + p_2\wedge p_3)}\right) \frac{\Gamma(-2sl_s^2)\Gamma(-2ul_s^2)}{\Gamma(1+2tl_s^2)} + \\ & + G_s \left(K_{tu} e^{i(p_1\wedge p_3 + p_2\wedge p_4)} + K'_{tu} e^{i(p_1\wedge p_3 + p_4\wedge p_2)}\right) \frac{\Gamma(-2tl_s^2)\Gamma(-2ul_s^2)}{\Gamma(1+2sl_s^2)}\,. \end{aligned} \tag{3.2}$$

The amplitude A_4 is obtained by integrating four vertex operators around the disc. We denote the two incoming particles by 1 and 2 and the two outgoing particles by 3 and 4 and let all momenta be incoming. Using Mobius invariance the vertex operators of particles 1, 2 and 3 can be put at three fixed points on the boundary of the disc-mapping it to the upper half plane these are usually taken to be $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$, respectively. The location of the vertex operator of particle number 4, z_4, is then integrated over the real axis. Since a Mobius transformation does not change the cyclic ordering of the vertex operators, we need to add another piece obtained from fixing $z_4 = 1$ and integrating the location of particle number 2. The three terms in the answer correspond to $-\infty < z_4 < 0$, $1 < z_4 < \infty$ and $0 < z_4 < 1$, respectively and similarly for $2 \leftrightarrow 4$. Here $p \wedge k = \theta^{01}(p_0k_1 - k_0p_1)$.

The kinematic factors K in (3.2) involve momenta, p_i, polarization vectors, ξ_i, and also traces over Chan Paton factors λ_i. The quantities s, t, u are the Mandelstam variables

$$s = 2p_1p_2\,, \qquad t = 2p_1p_4\,, \qquad u = 2p_1p_3 \tag{3.3}$$

satisfying the mass shell constraint $s+t+u = 0$. Scattering in the backward direction is defined by $u = 0$. In eq. (3.3) we used the open string metric to contract the indices.

For the case of backward scattering, $u = 0$, the kinematics are such that only the first term corresponding to the s-channel exchange gets multiplied by phases. One phase occurs when particles 1, 2 and 3 are placed at $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$, respectively and the location of particle 4 is integrated from $-\infty < z_4 < 0$. The opposite phase occurs when $2 \leftrightarrow 4$. No phases multiply the other two terms. One of the two phases, $e^{-2\pi i\tilde{E}sl_s^2}$, caused the appearance of the advanced waves in the non-commutative field model.

Setting u to zero, the first term in the amplitude takes the form

$$A_{st} \sim G_s\left(K_{st}e^{2\pi i\tilde{E}sl_s^2} + K'_{st}e^{-2\pi i\tilde{E}sl_s^2}\right)\Gamma(-2sl_s^2)\Gamma(2sl_s^2)\,. \tag{3.4}$$

Using the identity

$$y\Gamma(y)\Gamma(-y) = -\frac{\pi}{\sin(\pi y)}\,, \tag{3.5}$$

we can write this as follows:

$$A_{st} \sim G_s\left(K_{st}e^{2\pi i\tilde{E}sl_s^2} + K'_{st}e^{-2\pi i\tilde{E}sl_s^2}\right)\frac{1}{s\sin(2\pi sl_s^2)}\,. \tag{3.6}$$

The kinematic factors K are also simple in this case. They are proportional to s^2 times products of polarization vectors and traces over Chan Paton factors. Therefore,

$$A_{st} \sim G_s s\left(a_1e^{2\pi i\tilde{E}sl_s^2} + a_2e^{-2\pi i\tilde{E}sl_s^2}\right)\frac{1}{\sin(2\pi sl_s^2)}\,, \tag{3.7}$$

where the constants a_1 and a_2 are independent of s.

This term has poles at $s = n/2l_s^2$ with n being an integer. The divergence of the amplitude at the poles is an essential physical feature of the amplitude, a resonance corresponding to the propagation of an intermediate string state over long spacetime distances. To define the poles we use the correct ϵ prescription replacing $s \to s + i\epsilon$. This has the effect of shifting the poles off the real axis. Then the

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function $1/\sin(2\pi s l_s^2)$ can be expanded as a power series in $y = e^{2i\pi s l_s^2 - \epsilon}$. In all, this term in the amplitude takes the form

$$A_{st} \sim G_s s \sum_{n>0 \text{ odd}} a_1 e^{2\pi i(n+\tilde{E})sl_s^2} + a_2 e^{2\pi i(n-\tilde{E})sl_s^2} + O(\epsilon)\,. \tag{3.8}$$

Comparing with eq. (2.30), we see that the amplitude looks similar to the case of a non-commutative field theory. We get a sum of phases with the identification

$$\theta'_n = 2\pi(n \pm \tilde{E})l_s^2\,, \quad n > 0\,,\ \text{odd}\,. \tag{3.9}$$

What is interesting is that the non-commutativity parameter θ gets modified by stringy oscillator effects. In fact the phases persist even in the absence of the electric field. We see that θ'_n are positive for all positive odd integers.

In contrast with the field theory case, here we get phases that cause time delays only. The "acausal" phase, $e^{-2\pi i\tilde{E}s}$, gets multiplied by powers of y from the Gamma functions. The net effect is to produce phases which, for $\tilde{E} < 1$, or for $E < E_{\mathrm{cr}}$, cause time delays. Evidently, even in the presence of a background electric field, string scattering amplitudes produce causal behavior only. The acausal behavior due to the non-commutativity parameter θ is cancelled by phases from the Gamma functions. It seems that the oscillators are crucial for the causal behavior of the theory. The effects of the non-commutativity are always mixed with the effects of the string oscillators. We see another reason why space/time non-commutative field theories cannot be obtained as limits of string theory in background electric fields, as was found in [4, 5]. Such theories show pathological acausal behavior and are not unitary. What is interesting, however, is that the onset of the acausal behavior we found occurs as the electric field approaches its critical value.

The other two terms in formula (3.2) can be analyzed in a similar way. It is easier to write

$$A_{su} + A_{tu} = G_s(A_1 + A_2)\,, \tag{3.10}$$

where

$$A_1 \sim (K_{su} + K_{tu})\left(\frac{\Gamma(-2sl_s^2)\Gamma(-2ul_s^2)}{\Gamma(1+2tl_s^2)} + \frac{\Gamma(-2tl_s^2)\Gamma(-2ul_s^2)}{\Gamma(1+2sl_s^2)}\right) \tag{3.11}$$

and

$$A_2 \sim (K_{su} - K_{tu})\left(\frac{\Gamma(-2sl_s^2)\Gamma(-2ul_s^2)}{\Gamma(1+2tl_s^2)} - \frac{\Gamma(-2tl_s^2)\Gamma(-2ul_s^2)}{\Gamma(1+2sl_s^2)}\right). \tag{3.12}$$

Then we can analyze the sum and differences of the two combinations of Gamma functions that appear in (3.2) as $u \to 0$. Again, no Moyal phases multiply these two terms.

Setting $u = 0$, we find that

$$A_1 \sim a_3 s \frac{\cos(2\pi s l_s^2)}{\sin(2\pi s l_s^2)} \,. \tag{3.13}$$

Shifting $s \to s + i\epsilon$, we can expand $1/\sin(2\pi s l_s^2)$ in powers of y. Thus we find

$$A_1 \sim a_3 s \left(1 + e^{4\pi i s l_s^2}\right) \sum_{n \geq 0 \text{ even}} e^{2\pi i n s l_s^2} \,. \tag{3.14}$$

The first term in the series is just proportional to s and produces no large time delays. The other terms are phases responsible for time delays. The phases in this term are independent of θ. They are present even in the absence of a background field. Ordinary scattering of open strings shares features with scattering in non-commutative field theory with effective non-commutativity parameter the string length squared. However the amplitude produces only causal behavior.

The other term produces a pole at $u = 0$. We have that

$$A_2 \sim a_4 s \frac{1}{u l_s^2 + i\epsilon} \,. \tag{3.15}$$

The pole in u corresponds to the exchange of a massless particle and we will ignore it. This term has no oscillations in s.

The effect on the outgoing wave-packet is similar to the previous example except that the advanced waves are absent. Let us consider the case of no electric field for simplicity. Then $\theta'_n = 2\pi n l_s^2$ and the open string metric $G_{\mu\nu} = \eta_{\mu\nu}$. If we use eq. (2.23) for the same $\phi_{\rm in}(p)$ given in eq. (2.31), we find for a typical phase in the amplitude

$$\Phi_{\rm out}(x) \sim G_s \int dp p^2 e^{4i\theta'_n p^2} \left(e^{-(p-p_0)^2/\lambda} + e^{-(p+p_0)^2/\lambda}\right) e^{ipx} \,. \tag{3.16}$$

For $p_0 \gg \lambda^{1/2} \gg 1/2\pi p_0 l_s^2$, this is proportional to

$$\Phi_{\rm out}(x) \sim G_s \frac{d^2}{dx^2} F(x; \theta'_n, \lambda, p_0) + (p_0 \to -p_0) \tag{3.17}$$

with $F(x; \theta_n, \lambda, p_0)$ given by eq. (2.34).

We see that the outgoing wave-packet splits into a series of packets, one localized at $x = 0$, and a series at $x = 8p_0\theta'_n \;\; n > 0$. The advanced waves are absent. Only delayed waves are present. Each delayed packet has width given by $8\lambda^{1/2}\theta'_n$. The packets are not overlapping for $p_0 \gg \lambda^{1/2} \gg 1/2\pi p_0 l_s^2$. The n-th packets are more spread. After the derivatives are performed we find that the contributions to the sum are dominated by the small n packets. The amplitude of the packets falls like $n^{-5/2}$. Again the time delays are proportional to the energy p_0. It is interesting that the large time delays persist even in the absence of the electric field.

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The interpretation is different than before. The scattering is causal. We would like to suggest the following to explain the series of time delays. As the two strings come together, an intermediate stretched string state is formed. The string state has total energy p_0. The state is oscillating from small size to a large size proportional to p_0. To see this we write

$$p_0 = \frac{L}{l_s^2} + \frac{N}{L}\,, \tag{3.18}$$

where N is some oscillation number. We see that this is minimized for $L \sim p_0 l_s^2$. The state begins from small size and grows to a string of maximal size of order $p_0 l_s^2$, storing the energy as potential energy. This repeats itself periodically. With each oscillation there is an amplitude for the string to split. Thus there is an infinite sequence of delayed wave packets. The delay is proportional to L since the string ends move with the speed of light. The intermediate state has size proportional to the energy p_0. This is a manifestation of the stringy uncertainty relation.

In the case of a background electric field we find time delays (in closed string units) proportional to

$$\Delta t = \frac{p_0 l_s^2}{1 - \tilde{E}^2}\,. \tag{3.19}$$

The time delays are proportional to $1/T_{\mathrm{eff}}$, where $T_{\mathrm{eff}} = (1 - \tilde{E}^2)/l_s^2$ is the effective tension of the open strings in the presence of the electric field. The effect of the electric field is to reduce the tension of the strings [4, 14]. As the electric field approaches its critical value, the time delays become longer. The extent of the intermediate state in space is also bigger. However, we note that as the field approaches its critical value, the effective coupling constant G_s tends to zero and the amplitudes are suppressed.

We have illustrated the violations of causality in space/time non-commutative field theory and its restoration in string theory by considering the evolution of wave packets. Evidently the scattering amplitudes of the field theory violate some principle of S-matrix theory that string theory preserves. In fact it is not difficult to see what principle is involved. Macroscopic causality is usually assumed to follow from two properties of amplitudes. The first involves the location of singularities in the Mandelstam s variable; namely, the amplitude should be analytic in the upper half plane. In the case of non-commutative field theory the amplitude in eq. (2.30) is an entire function and satisfies this rule. In the case of the string theory tree diagrams there is an infinite sequence of poles on the real axis. However with the conventional $i\epsilon$ prescription the poles are displaced to the lower half plane and lead to no violation of causality. The second requirement is that the amplitudes should not exponentially diverge along any direction in the upper half plane in order to insure that certain contours of integration can be closed. This is what is violated in the non-commutative field theory. The cosine term in eq. (2.30) exponentially diverges in the upper half plane. By contrast the non-commutative Moyal phases in string theory are compensated for by the factor $1/\sin(2\pi s l_s^2)$ in formulae like eq. (3.7) as long as the electric field is smaller than critical.

4. Conclusion

Space/time non-commutativity is a more subtle phenomenon than its space/space counterpart. If we define the space/time non-commutative deformation of a theory by multiplying its tree diagrams by space/time Moyal phases then an ordinary quantum field theory becomes acausal as well as non unitary. The acausality is easily seen in the scattering of wave packets by the appearance of an outgoing signal that originates before the incoming particles reach each other.

By contrast, the space/time non-commutative deformation of open string theory is not acausal. The theory does not have a limit in which stringy effects disappear. These stringy effects conspire to shift the Moyal phases so that they become causal. Thus the peculiar advanced effects found in the field theory should not be though of as the signature of non-commutativity.

The delayed effects of non-commutativity in a collision process are also interesting. The space/time non-commutativity manifests itself by time delays which grow linearly with increasing momentum. As we have seen in section 3 the time delay of the leading delayed wave is governed by a parameter $\theta'_{\pm} = 2\pi(1 \pm \tilde{E})l_s^2$. Nothing special seems to happen to θ' as the electric field is turned off. However θ'_{-} vanishes at the critical electric field. One possible interpretation of this is that open string theory exhibits the signature of space/time non-commutativity without any electric field. Indeed this interpretation is suggested by the well known space/time uncertainty principle [10, 11, 12], i.e. it appears like the string grows in the longitudinal direction to a length of order pl_s^2.

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